Robust $H_\infty$ Polynomial Fuzzy Controller Design: An Sum-of-square-based Quantum-inspired Evolutionary Algorithm Approach

Gwo-Ruey Yu, IEEE Member, Yu-Chia Huang, Chih-Yung Cheng

Abstract—This paper proposes a robust $H_\infty$ polynomial fuzzy controller design by an sum-of-squares-based (SOS-based) quantum-inspired evolutionary algorithm (QEA) approach for a polynomial fuzzy system with model uncertainties and external disturbances. QEA is adopted to evolve optimal control gains with a fitness function that is defined by performance requirements. The stability and robustness of the control system are guaranteed by the proposed robust $H_\infty$ stability conditions, which are formed by the sum-of-squares (SOS) method. Simulation results demonstrate the effectiveness of the proposed approaches.

I. INTRODUCTION

In 2009, Tanaka et al. [1] proposed a polynomial fuzzy controller design with sum-of-squares (SOS) approach. In [1], a polynomial fuzzy control system is a general framework of a Takagi-Sugeno (T-S) fuzzy control system because the system states are allowed to exist in system matrices and control gains. In stability analysis, a polynomial Lyapunov function is used instead of quadratic Lyapunov function to derived stability conditions by SOS approach. Using SOSTOOLS [2] in MATLAB, control gains are obtained by solving the stability conditions. Recently, there are many extended researches about polynomial fuzzy control [3] – [9].

On the other hand, a physical plant cannot be modeled accurately because of parameter variations, unmodeled dynamics and unexpected disturbances. To deal with external disturbances and model uncertainties, there are many studies which are investigated with these problems [10] – [12]. For instance, a design of robust controller was proposed to guarantee a stability of an uncertain system [10]. To reject external disturbances, $H_\infty$ performance and disturbance observer were used to design polynomial fuzzy controllers [11]. A design framework for robust control of polynomial fuzzy systems was proposed to deal with model uncertainties. Furthermore, a robust $H_\infty$ control design for polynomial fuzzy systems, where model uncertainties and external disturbances are simultaneously considered, was proposed in [12].

In these studies, control gains can be obtained only by solving SOS-based stability conditions. However, specific performance requirements cannot be met in stability conditions. To solve this problem, a design procedure called SOS-based quantum-inspired evolutionary algorithm (SOS-based QEA) was proposed in [13]. The SOS-based QEA applied to search for control gains with the best performance. Then, the stability of the closed-loop system is guaranteed under SOS-based stability conditions. However, the stability conditions discussed in [13] are used to only guarantee the stability of polynomial fuzzy systems which do not have model uncertainties and external disturbances. Therefore, this paper proposes a robust $H_\infty$ polynomial fuzzy controller design by SOS-based QEA approach for a polynomial fuzzy system with model uncertainties and external disturbances. To enlarge the search space, relaxed SOS-based robust $H_\infty$ stability conditions are derived based on a principle of copositivity.

The remainder of this dissertation is organized as follows. Section II introduces the polynomial fuzzy control systems and the SOS-based QEA. Section III proposed SOS-based robust $H_\infty$ stability conditions and relaxed SOS-based robust $H_\infty$ stability conditions. Section IV offers conclusions.

II. POLYNOMIAL FUZZY CONTROL SYSTEMS

A. Continuous-time Polynomial Fuzzy Control Systems

Consider a continuous-time nonlinear system:

$$\dot{x}(t) = f(x(t), u(t), d(t)), \quad y(t) = g(x(t)), \tag{1}$$

where $f(x(t), u(t), d(t))$ and $g(x(t))$ are nonlinear functions, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^r$ is the output vector, and $d(t) \in \mathbb{R}^q$ is the disturbance input with bounded $L_2$ norm, that is, $d(t) \in L_2[0, \infty)$. By the concept of sector nonlinearity, a polynomial fuzzy model constructed with model uncertainties and external disturbances is presented to represent (2).

$$\dot{x}(t) = \sum_{i=1}^{m} h_i(z(t))\left[ A_i(x(t)) + W_{i}^r(x(t))A_{i}^r(t)Y_{i}^r(x(t)) \right] \hat{x}(x(t)) + \{ B_j(x(t)) + W_{j,i}(x(t))A_{j,i}(t)Y_{j,i}(x(t)) \} u(t) + E_j(x(t))d(t)$$

$$y(t) = \sum_{i=1}^{m} h_i(z(t))C_i(x(t)) \hat{x}(x(t)), \tag{2}$$

where $\sum_{i=1}^{m} h_i(z(t)) = 1$, $h_i(z(t)) = w_i(z(t)) \prod_{j=1}^{r} \mu_{ij}(z_j(t))$ and $\mu_{ij}(z_j(t))$ is the grade of membership of $z_j(t)$ in $M_i$ for $i = 1, 2, \cdots, r$. $r$ is the number...
of the rules, \( p \) is the number of the premise variables, \( z_i(t), \ldots, z_p(t) \) are the premise variables and 
\( z(t) = [z_1(t), \ldots, z_p(t)]^T \). \( M_{ij} \) is the fuzzy set. \( \hat{x}(t) \in \mathbb{R}^N \) is a vector whose entries are all monomials in \( x(t) \). A monomial is a function that form as 
\( x_1^{\xi_1}(t)x_2^{\xi_2}(t)\cdots x_n^{\xi_n}(t) \), 
where \( \xi_1, \xi_2, \ldots, \xi_n \) are nonnegative integers. 
\( A_i(x(t)) \in \mathbb{R}^{n \times N}, \ B_i(x(t)) \in \mathbb{R}^{n \times m}, \ C_i(x(t)) \in \mathbb{R}^{n \times N}, \ E_i(x(t)) \in \mathbb{R}^{m \times n}, \ W_{ai}(x(t)) \in \mathbb{R}^{n \times m}, \ Y_{ai}(x(t)) \in \mathbb{R}^{n \times n}, \ W_{bi}(x(t)) \in \mathbb{R}^{m \times m} \), 
\( Y_{bi}(x(t)) \in \mathbb{R}^{m \times n} \) are polynomial matrices. \( \Delta_{ai}(t) \in \mathbb{R}^{n \times q} \) and \( \Delta_{bi}(t) \in \mathbb{R}^{m \times w} \) are uncertain blocks which satisfy: 
\[
\begin{align*}
\| \Delta_{ai}(t) \| & \leq \frac{1}{\eta_{ai}}, \quad \Delta_{ai}(t) = \Delta_{ai}^0(t), \\
\| \Delta_{bi}(t) \| & \leq \frac{1}{\eta_{bi}}, \quad \Delta_{bi}(t) = \Delta_{bi}^0(t).
\end{align*}
\]

By the parallel distributed compensation (PDC) principle, a polynomial fuzzy controller shares the same membership function and number of rules as the given polynomial fuzzy model. The output of the controller is 
\[
u(t) = -\sum_{i=1}^{p} h_i(z(t))F_i(x(t))\hat{x}(x(t)),
\]
where \( F_i(x(t)) \in \mathbb{R}^{m \times n} \) is the control gain. The overall of the closed-loop polynomial fuzzy control system is 
\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{p} \sum_{j=1}^{q} h_i(z(t))h_j(z(t))\left( [A_i(x(t)) + B_i(x(t))F_i(x(t)) + W_{ai}(x(t))\Delta_{ai}(t)Y_{ai}(x(t)) + W_{bi}(x(t))\Delta_{bi}(t)Y_{bi}(x(t))F_i(x(t)) + E_i(x(t))d(t) \right)h_j(z(t))C_j(x(t))\hat{x}(x(t)), \\
y(t) &= \sum_{i=1}^{p} h_i(z(t))C_i(x(t))\hat{x}(x(t)),
\end{align*}
\]

Remark 1: If \( \dot{x}(x(t)) = x(t) \) and \( A_i(x(t)), B_i(x(t)), C_i(x(t)), E_i(x(t)), W_{ai}(x(t)), W_{bi}(x(t)), Y_{ai}(x(t)), Y_{bi}(x(t)), \) and \( F_i(x(t)) \) are constant matrices, the polynomial fuzzy control system (6) will be deemed as a continuous-time T-S fuzzy control system.

B. SOS-based QEA

SOS-based QEA is a synthesis of the QEA and SOS-based stability conditions [20]. The aim of the SOS-based QEA is to enable a systematic design of a polynomial fuzzy controller which simultaneously guarantees system stability and achieves optimal performances. The procedure of the SOS-based QEA is shown in Fig. 1.

III. ROBUST H- POLYNOMIAL FUZZY CONTROLLER DESIGN

A. SOS-based Robust H–Stability Conditions

To deal with external disturbances in the polynomial fuzzy control system (6), the disturbance rejection problem can be realized in the following inequality:
\[
\sup_{x(t) \in \mathbb{R}^N} \left\{ \frac{\| y(t) \|}{\| d(t) \|_2} \right\} \leq \gamma
\]
where \( \gamma \) denotes an index of \( H_\gamma \) performance. To simplify the representations, symbols \( x, y, d, \Delta_{ai}, \) and \( \Delta_{bi} \) are employed instead of \( z(t), x(t), y(t), d(t), \Delta_{ai}(t), \) and \( \Delta_{bi}(t) \) in this section.

Theorem 1: Assume \( A_i(x), B_i(x), E_i(x), W_{ai}(x), W_{bi}(x), Y_{ai}(x), Y_{bi}(x), \) and \( F_i(x) \) are known, the stability of the continuous-time polynomial fuzzy control system (6) can be guaranteed if there exists a positive definite polynomial matrix \( P(x) \) such that (14) and (15) are satisfied.
\[
\begin{align*}
\tilde{V}(P(x) - \varepsilon_1(x)I) \Theta(x) & = \text{SOS}, \\
\tilde{V}^2 \left[ \Phi_{ij}(x) - \frac{1}{2} \Omega_{ij}(x) \Omega_{ij}(x)^* - \varepsilon_{2ij}(x)I \right] \Phi_{ij} & = \text{SOS},
\end{align*}
\]
where \( \varepsilon_{1}(x) > 0 \) and \( \varepsilon_{2ij}(x) \geq 0 \) are polynomials for all \( x \), while \( \gamma, \eta_{ai}, \) and \( \eta_{bi} \) are previously given nonnegative parameters. \( H(x) \in \mathbb{R}^{N \times N} \) is a polynomial matrix whose \( (i, j) \)-th entry is given by 
\[
H_{ij}(x) = \frac{\partial \hat{x}_{ij}(x)}{\partial x_j}.
\]
\[ G_j(x) = -A^T_j(x)H^T(x)P(x) + F_j^T(x)B_j^T(x)H^T(x)P(x) \\
- P(x)H(x)A_j(x) + P(x)H(x)B_j(x)F_j(x) - C_j^T(x)C_j(x) \]
\[ - \sum_{k=1}^{\infty} \frac{\partial P(x)}{\partial x_k} \left( A_i^T(x) - B_i^T(x)F_j(x) \right) \hat{x}(x) \]
\[ \Theta_j(x) = \left[ \begin{array}{c} G_{j}(x) + G_{\mu}(x) \\ \frac{1}{2} \left( E_{j}^T(x) + E_{j}^T(x)H(x)Q(x) \right) \\
- \frac{1}{4} \sum_{i=1}^{n} \left( E_{i}^T(x) + E_{i}^T(x) \hat{x}(x) \right) \frac{\partial Q(x)}{\partial x_i} \end{array} \right] y^2 \]
\[ \Gamma_{ij}(x) = \left[ \begin{array}{c} \left( I \otimes W_{ij}^a \right)^T \frac{\partial P(x)}{\partial x_i} \\ \left( I \otimes W_{ij}^b \right)^T \frac{\partial P(x)}{\partial x_j} \end{array} \right], \quad \Omega_{ij}(x) = \left[ \begin{array}{c} Y_{ij}(x) \\ 0 \\ \cdots \\ 0 \end{array} \right] \]
\[ S_{ij}(x) = \left[ \begin{array}{c} I \otimes (Y_{ij}(x) \hat{x}(x)) \\ \cdots \\ I \otimes (Y_{ij}(x) \hat{x}(x)) \end{array} \right], \quad \bar{S}_{ij}(x) = \left[ \begin{array}{c} I \otimes (Y_{ij}(x)F_j(x) \hat{x}(x)) \\ \cdots \\ I \otimes (Y_{ij}(x)F_j(x) \hat{x}(x)) \end{array} \right] \]

The proof is omitted due to the limitation of paper length.

**B. Numerical Example 1**

Consider the following nonlinear system with model uncertainties and external disturbances [19]:
\[ \dot{x}_1 = -x_1 + x_1^2 + x_2^3 + x_1 x_2 - x_1 x_2^2 + x_2 + x_1 \mu + (0.1 x_1^2 + 0.1 \sin(x_1)) \Delta + (0.001 + 0.1 x_1) \Delta \mu \\
+ (0.01 x_1 + 0.1) d \]
\[ \dot{x}_2 = -\sin(x_1) - x_2 \]
\[ y = 0.01 x_1^2 + 0.1 x_1 \]
where \( x_1 \) and \( x_2 \) denote the system states, \( y \) denotes the system output, \( \Delta \) denotes the uncertain blocks, and \( d \) denotes the disturbance signal. Let \( z = \sin(x_1)/x_1 \) be the premise variable, then the polynomial fuzzy model can be obtained as:
\[ \dot{x} = \sum_{i=1}^{n} h_i(z) \left( A_i(x) + W_{ai}(x) \Delta_i + Y_{ai}(x) \right) \hat{x}(x) \]
\[ + \left( B_i(x) + W_{bi}(x) \Delta_i + Y_{bi}(x) \right) u + E_i(x) d \]
\[ y = \sum_{i=1}^{n} h_i(z) C_i(x) \hat{x}(x), \]
where \( \hat{x}(x) = x = [x_1 \quad x_2]^T \)

\[ h_i(z) = \frac{x_i - \sin(x_i)}{1.2172 x_i}, \quad h_i(z) = \frac{\sin(x_i) + 0.2172 x_i}{1.2172 x_i}, \]
\[ A_i(x) = \left[ \begin{array}{c} -1 + x_1 + x_1^2 + x_2 + x_2^2 - 2 \\ 0.2172 \end{array} \right], \]
\[ A_i(x) = \left[ \begin{array}{c} -1 + x_1 + x_1^2 + x_2 + x_2^2 - 2 \\ -1 \end{array} \right] \]
\[ B_i(x) = B_i(x) = [x_1, 0]^T, \]
\[ E_i(x) = E_i(x) = [0.01 x_1 + 0.1, 0]^T, \]
\[ C_i(x) = C_i(x) = [0.01 x_1 + 0.1, 0], \]
\[ W_{ai}(x) = [x_1 - 0.2172, 0]^T, \]
\[ W_{bi}(x) = [x_2 + 1, 0]^T \]
\[ Y_{ai}(x) = Y_{ai}(x) = [0.1, 0] \]
\[ Y_{ai}(x) = Y_{ai}(x) = [0.1 + x_1, 0]^T \]
\[ \Delta_{bi} = \Delta_{ai} = \Delta_{bi} = \Delta \]

The objective of this example is to design a controller such that the system states will converge to zero. The SOS-based stability conditions proposed by [19] offered control gains with \( \gamma_{\min} \) and \( \eta_{\min} \), described as:
\[ \gamma_{\min} = 0.1406, \quad \eta_{\min} = 0.3307 \]
\[ Q(x) = \left[ \begin{array}{c} 1.5416 -0.0000 \\ -0.0000 1.5415 \end{array} \right] \]
\[ F_1(x) = [0.6923 -0.0666 + 4.7057 0.0003] x_1 \\
+ [0.5500 -0.0000] x_2 \]
\[ F_2(x) = [1.0632 -0.0849 + 4.7518 0.0003] x_1 \\
+ [0.5396 -0.0000] x_2 \]

**TABLE I. THE PARAMETERS OF THE ROBUST \( H_\infty \) SOS-BASED QEA.**

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of iteration generation</td>
<td>30</td>
</tr>
<tr>
<td>Number of population</td>
<td>16</td>
</tr>
<tr>
<td>Length of qubit</td>
<td>14</td>
</tr>
<tr>
<td>( \mu )</td>
<td>0.01 ( \pi )</td>
</tr>
</tbody>
</table>

**TABLE II. RANGE OF FEEDBACK GAINS.**

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Value</th>
<th>Quantity</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{11} )</td>
<td>-7 -7</td>
<td>( f_{31} )</td>
<td>-7 -7</td>
</tr>
<tr>
<td>( f_{21} )</td>
<td>-1 -1</td>
<td>( f_{22} )</td>
<td>-1 -1</td>
</tr>
<tr>
<td>( f_{31} )</td>
<td>-30 -30</td>
<td>( f_{32} )</td>
<td>-30 -30</td>
</tr>
<tr>
<td>( f_{41} )</td>
<td>-0.002 -0.002</td>
<td>( f_{42} )</td>
<td>-0.002 -0.002</td>
</tr>
<tr>
<td>( f_{51} )</td>
<td>-4 -4</td>
<td>( f_{52} )</td>
<td>-4 -4</td>
</tr>
<tr>
<td>( f_{61} )</td>
<td>-0.000 -0.0000</td>
<td>( f_{62} )</td>
<td>-0.000 -0.0000</td>
</tr>
</tbody>
</table>

With the same values of \( \gamma_{\min} \) and \( \eta_{\min} \), the proposed robust \( H_\infty \) SOS-based QEA is employed to seek feasible and optimal solutions in Theorem 1. Tab. I lists the parameters of the robust \( H_\infty \) SOS-based QEA. To obtain the optimal control performance, the fitness function is defined as the integral absolute error (IAE):
\[ IAE = \int (|x_1| + |x_2|) dt \]
The feedback gains are defined as:
\[ F_1(x) = [f_{11}, f_{12}] + [f_{13}, f_{14}]x_1 + [f_{15}, f_{16}]x_2, \]
\[ F_2(x) = [f_{21}, f_{22}] + [f_{23}, f_{24}]x_1 + [f_{25}, f_{26}]x_2, \]
where the range of \( f_{11}, f_{12}, \ldots, f_{16}, f_{21}, f_{22}, \ldots, f_{26} \) is listed in Tab. II. By minimizing the fitness function, the control gains are evolved. Since the algorithm is stochastic, the experiments were performed one thousand times to obtain the optimal solution. Fig. 2 shows the progress of the average fitness. Finally, the optimal feedback gains are sought out.

\[ F_1(x) = [1.6229, -0.2232] + [13.8825, 0.0005]x_1 + [2.6009, 0.0001]x_2 \]
\[ F_2(x) = [0.1330, -0.0358] + [15.2725, 0.0010]x_1 + [-2.0845, 0.0001]x_2 \]

With the initial conditions \( x(0) = [2, 0]^T \) and the uncertain blocks and external disturbances, defined as:

\[ d = \begin{cases} 5, & 2 \leq t \leq 3; \\ 0, & \text{other}. \end{cases} \]
\[ \Delta = 2 \sin(\pi t). \]

Fig. 3 and 4 show the system responses of system states, and the control performances is shown in Fig. 5.

**Summary 1:** According to the simulation results, even if the control gains are obtained by solving the existing SOS-based stability conditions with minimum \( \gamma \) and \( \eta \), the control performance may not be satisfied. However, using the same robust parameters \( \gamma \) and \( \eta \), the proposed robust \( H_{\infty} \) SOS-based QEA could find optimal solutions, and the optimally robust \( H_{\infty} \) polynomial fuzzy controller is achieved.

**C. Relaxed SOS-based Robust \( H_{\infty} \) Stability Conditions**

Lemma 1 presents the definition of copositivity. However, checking the copositivity of the matrix is a co-NP problem. To reduce the difficulty, a relaxed approach is utilized to guarantee the copositivity of the matrix, which is described in Lemma 2. Then, Lemma 2 and Lemma 3 are used to derive the relaxed SOS-based robust \( H_{\infty} \) stability conditions, which are stated in Theorem 2.

Lemma 1 [14]: If the following equation holds, the matrix \( G = [G_{ij}] \in \mathbb{R}^{n \times n} \) is copositive:

\[ h^T G h = \sum_{i=1}^{n} \sum_{j=1}^{n} h_i h_j G_{ij} \geq 0 \]

where \( h = [h_1, h_2, \ldots, h_n]^T \in \mathbb{R}^n \) and \( h_i \geq 0 \).
Lemma 2 [14]: There is a relaxation for Lemma 1 if the following inequality is satisfied:

\[ f^\alpha(s) = \left( \sum_{i=1}^{n} s_i^2 \right)^\alpha \sum_{i=1}^{n} s_i^2 G_{ij} \text{ is SOS} \]  

where \( s = [s_1, s_2, \ldots, s_n]^T \) and \( \alpha \) is a nonnegative integer.

**Theorem 2**: Assuming \( A_i(x), B_i(x), E_i(x), W_{ai}(x), W_{bi}(x), Y_{ai}(x), Y_{bi}(x) \), and \( F_j(x) \) are known, the stability of the continuous-time polynomial fuzzy control system (6) can be guaranteed if there exist a positive polynomial Lyapunov function \( V(x) \) and a polynomial \( \theta_i(x) \) such that (19) – (21) are satisfied with \( \xi_{ai} > 0 \) and \( \xi_{bi} > 0 \):

\[ V(x) - \xi_i(x) \text{is SOS} \]  

\[ \left( \sum_{i=1}^{n} s_i^2 \right)^\alpha \sum_{i=1}^{n} s_i^2 G_{ij} \]  

\[ \begin{aligned} 
\frac{\partial V(x)}{\partial x} &\left( -A_i(x) + B_i(x)F_j(x) \right) \dot{x}(x) - E_i(x) = \\
- \dot{x}(x)^T C_i^T(x) C_j(x) \dot{x}(x) = \\
- \frac{1}{2} D^{T}_{ij}(x) \Psi D_{ij}(x) - \theta_i(x) + \gamma^2 d^T d \end{aligned} \text{is SOS} \]  

\[ v^T \Omega(x) v \text{ is SOS} \]  

where \( v \) is a vector which is independent of \( x \); \( \xi_i(x) \) is a positive polynomial; \( \alpha \) is a nonnegative integer, and \( \gamma \), \( \eta_{ai} \) and \( \eta_{bi} \) are previously given nonnegative parameters.

\[ D_{ij}(x) = \begin{bmatrix} \eta_{ai} Y_{ai}(x) \dot{x}(x) \\ \eta_{bi} Y_{bi}(x) \dot{x}(x) \end{bmatrix}, \quad \Omega_i(x) = \begin{bmatrix} \theta_i(x) & R_i(x) \\ R_i^T(x) & 2 \Psi_i \end{bmatrix} \]  

\[ R_i(x) = \begin{bmatrix} - \frac{\partial V(x)}{\partial x} W_{ai}(x) \\ - \frac{\partial V(x)}{\partial x} W_{bi}(x) \end{bmatrix} \]  

\[ \Psi_i = \text{block-diag}(\xi_{ai}^2 I, \xi_{bi}^2 I) \]

The proof is omitted due to the limitation of paper length.

**D. Numerical Example 2**

Using the nonlinear system (10) in numerical example 1, with the same value of \( \gamma_{min}, \eta_{min} \), fitness function, and parameters, the SOS-based QEA is employed to seek feasible and optimal solutions in Theorem 2 (\( \alpha = 0 \)). By minimizing the fitness function the following is obtained:

\[ F_1(x) = [1.4063, -0.2628] + [4.8667, -0.0007] x_1 \\
+ [-1.2107, -0.0000] x_2 \]  

\[ F_2(x) = [5.9221, -0.1918] + [27.5351, -0.0007] x_1 \\
+ [0.8587, -0.0000] x_2 \]  

Fig. 6 shows the progress of the average fitness. Fig. 7 – 9 show the system responses and IAE with the initial conditions \( x(0) = [2 \ 0]^T \), the external disturbances (15), and uncertain blocks (16).

Summary 2: The controller design by Theorem 2 exhibits better performance than the controller design by Theorem 1 because the stability conditions in Theorem 2 is more relaxed such that they will be more easily satisfied with optimal solutions.
E. Numerical Example 3

This example demonstrates the difference in feasible area between Theorem 1 and Theorem 2. Consider a two-rule continuous-time polynomial fuzzy model where the system matrices are as follows:

\[ A_1(x) = \begin{bmatrix} 0 & 1 \\ 17.2941 & 0 \end{bmatrix}, \quad A_2(x) = \begin{bmatrix} 0 & 1 \\ 9.36 & 0 \end{bmatrix} \]

\[ B_1(x) = \begin{bmatrix} 1 & 0 \\ 0 & -0.17647 \end{bmatrix}, \quad B_2(x) = \begin{bmatrix} 1 & 0 \\ 0 & -0.005235881 \end{bmatrix} \]

\[ E_1(x) = E_2(x) = [0 \ 0.01]^T, \quad C_1(x) = C_2(x) = [0.01 \ 0] \]

\[ W_{a_1}(x) = W_{a_2}(x) = [0 \ 0.01]^T, \quad Y_{a_1}(x) = Y_{a_2}(x) = [0 \ 0.01] \]

\[ W_{b_1}(x) = W_{b_2}(x) = [0 \ 0.01]^T, \quad Y_{b_1}(x) = Y_{b_2}(x) = [0 \ 0.01] \]

The parameters of the SOS-based QEA are listed in Tab. I. In the process of finding the control gains, Theorem 1 and Theorem 2 are used to determine the control system stability with QEA-based gains. In Theorem 1, it is assumed that \( p(x) \) is constant. In Theorem 2, it is assumed that \( V(x) \) and \( \theta(x) \) are order-two polynomial, and \( \alpha = 0 \). The structure of the control gains is defined as:

\[ F_1(x) = \begin{bmatrix} f_{1a} & f_{1b} \\ f_{1c} & f_{1d} \end{bmatrix}, \quad F_2(x) = \begin{bmatrix} f_{2a} & f_{2b} \\ f_{2c} & f_{2d} \end{bmatrix} \]

(22)

where the range of \( f_{1a}, f_{1b}, \ldots, f_{1d}, f_{2a}, f_{2b}, \ldots, f_{2d} \) is between \(-3000\) to \(3000\). Assume \( \gamma = n_{a1} = n_{a2} = n_{b1} = n_{b2} = 1 \) in both theorems. Since the algorithm is stochastic, the experiments are conducted five hundred times, and the feasible solution area is plotted in Fig. 10. The points plotted in Fig. 10 indicate that the gains are satisfied with Theorem 2 but not satisfied with Theorem 1. Tab. III provides the number of feasible solutions by different theorems.

\[
\begin{array}{c|c|c}
\text{Maximum number of feasible solutions in one experiment} & \text{Theorem 1} & \text{Theorem 2} \\
\hline
\text{Number of feasible solutions with} & 28 & 31 \\
\text{500 times experiments} & 8836 & 7785 \\
\end{array}
\]

IV. CONCLUSIONS

This paper proposes a novel theorems for the design of robust \( H_\infty \) polynomial fuzzy controllers for polynomial fuzzy systems with SOS-based QEA. In the procedure of the proposed methods, the optimal solutions can be sought out by SOS-based QEA with performance requirements, and the stability and robustness of the control system are guaranteed by the proposed SOS-based robust \( H_\infty \) stability conditions. Simulation results demonstrated that the theorem with SOS-based QEA’s ability to find the optimal solution is superior to current approaches.

REFERENCES


