### Positive systems analysis

- Quadratic forms are widely used for system analysis. Lyapunov inequality, Kalman-Yakubovich-Popov Lemma, integral quadratic constraints, etc.
- Analysis can be simplified if systems are known to be positive
- Lyapunov inequality:
  - \[ AP + PA + Q > 0 \]
  - \( A \in \mathbb{R}^{n \times n}, \) \( P \in \mathbb{R}^{n \times n} \) are symmetric matrices, \( Q \in \mathbb{R}^{n \times n} \)
  - \( A \) is a stable matrix if \( A^T P + P A + Q > 0 \)
- Kalman-Yakubovich-Popov Lemma:
  - \[ \left| \frac{(A - D)^+ B C}{A^T} \right|^2 > 0 \quad \forall \omega \in \mathbb{R} \]
- \( A \) is a stable matrix if \( A^T (A - D)^+ B C > 0 \)
- The theory of integral linear constraints (ILC)

### Positive systems

A system \( G \) is said to be positive if

\[ (u(t), y(t)) \geq 0 \quad \forall t \geq 0 \]

Given a positive feedback interconnection of two positive systems \( G_1 \) and \( G_2 \), the closed-loop map \( (d_1, d_2) \mapsto (u_1, y_2, y_2) \) is always positive?

### Feedback interconnections

- Suppose (nonlinear) \( G_i \) are causal and positive, define instantaneous gain of \( G_i \)

\[ a(G_i) = \sup_{x \in X} \frac{\max_{y \in Y} y}{\max_{z \in Z} z} \]

Positivity of closed-loop map: If \( a(G_i) < 1 \), then \( (d_1, d_2) \mapsto (u_1, y_2, y_2) \) is positive

### Robust stability of positive feedback systems

#### Integral quadratic constraints (IQCs) [Megretski & Rantzer 97]

Given bounded, causal, linear \( G_1 : L_{2}^0 \to L_{2}^0 \) and \( G_2 : L_{2}^0 \to L_{2}^0 \), suppose there exists a linear (time-varying) \( Z \) such that

\[ \begin{align*}
(1) & : G_1(z(t)) G_2(z(t)) \\
(2) & : G_1(y(t)) G_2(y(t)) \\
(3) & : G_1(u(t)) G_2(u(t)) \\
(4) & : G_1(w(t)) G_2(w(t))
\end{align*} \]

then \( (G_1, G_2) \) is stable

### Geometric interpretation of integral linear constraints

#### Integral linear constraints

Given bounded, causal linear \( G_1 : L_{2}^0 \to L_{2}^0 \) and \( G_2 : L_{2}^0 \to L_{2}^0 \), suppose there exists linear (time-varying) \( Z \) such that

\[ \begin{align*}
(1) & : G_1(z(t)) G_2(z(t)) \\
(2) & : G_1(y(t)) G_2(y(t)) \\
(3) & : G_1(u(t)) G_2(u(t)) \\
(4) & : G_1(w(t)) G_2(w(t))
\end{align*} \]

then \( (G_1, G_2) \) is stable in \( L_{2}^0 \)

### A simple academic example...

- \( a(G_1) \) is a constant feedback delay \( (G_1, G_2) \) is stable.
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### Conclusions

- Extended the recently proposed ILC stability theorem for the positive feedback systems to the setting where component systems could be nonlinear and the multiplier could be time-varying.
- Developed some ILC’s for some uncertainties (more to come...)
- Demonstrated that time-varying ILC’s could be useful when the component systems are not LTI.
- Proposed extensions to “sign preserving” and “cone preserving” feedback systems.