Polynomial-based $H_{\infty}$ Observed-state Feedback Stabilization via Homogeneous Lyapunov Functions

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Abstract

This paper introduces a homogeneously polynomial Lyapunov function for a stabilization problem involving $H_{\infty}$ observed-state feedback control via fuzzy polynomial systems which contains the quadratic Lyapunov function as a special case. A new sufficient SOS condition formulated in terms of state dependent matrix inequalities to achieve the stabilization are proposed. The proposed scheme eliminates non-convex terms in the existing papers. Hence the SOS decomposition proposed in this paper is more general than the SOS-based methodologies to T-S fuzzy model control systems. To verify the analytic theories regarding polynomial fuzzy stabilization with the proposed method, two examples are demonstrated to show the methodology of the proposed approach.

Keywords: Polynomial TS Fuzzy Models, Homogeneous Lyapunov function, Sum Of Squares (SOS).

I. Introduction

Polynomial fuzzy systems whose consequence parts consist of polynomial state variables are extended from classical TS fuzzy systems whose consequence parts are constant matrices [1], [2], [3], [4]. Thanks to [5], polynomial fuzzy system can model exactly a given nonlinear systems with less fuzzy rules when compared to the classical TS fuzzy systems. However, the involvement of state variables in the fuzzy systems and controller/observer gain matrices render difficulties in stability/stabilization analysis via LMI [6]. To tackle the issue, stability conditions via sum-of-squares (SOS) has gained its momentum [7], [8]. And polynomial non-quadratic Lyapunov function is a natural extension to the constant quadratic Lyapunov function when polynomial fuzzy systems is used. However, existing approaches via polynomial non-quadratic Lyapunov function for the continuous-time fuzzy systems involves time-derivative of Lyapunov matrix and the coupling effect of the input matrices has to be annihilated for the controller design [8], [3].

Recently an SOS method to polynomial fuzzy controller-observer designs was introduced in [7] in which state dependent SOS stabilization and estimation conditions were derived and discussed for three different assumptions on the structure of systems matrices, yet only constant Lyapunov $P$ matrix was assumed for the augmented controller-observer system.

In the present paper, it is shown that semidefinite programming and the sum of squares decomposition can be also used for the case of homogeneous Lyapunov functions. This is established using Euler’s homogeneity relation for positive homogeneous functions. With that said, the solution to the proposed stability conditions can be found numerically with the third-party MATLAB toolbox SOSTOOLS [9].

The organization of this paper is as follows. Following the introduction, Section II will rehearse some key notions of classical TS models, working lemmas and theorems, characterized by homogeneous Lyapunov method. Section III introduces polynomial fuzzy-model-based system and the relevant Lyapunov results, establishing the theory applicable to T-S fuzzy systems. Section IV is devoted to numerical simulations, demonstrating protruding advantages compared to existing results. Concluding remarks are made in Section V.

Notations: $P(x) > 0$ denotes a symmetric matrix function $P(x) : R^n \rightarrow R^{n\times n}$ which is positive definite for all $x \in R^n$. Column vector $\nabla V(x) = \frac{\partial V}{\partial x}(x)$ denotes the derivative of $V$ with respect to $x$ and $\nabla_{xx}V(x)$ denotes the Hessian of $V$. The time derivative of $V$ along the vector field $f : R^n \rightarrow R^n$ is denoted by $\nabla V(x)^T f(x)$. Also, $\cdot^T$ denotes transpose operation. The notation $Y_i$ stands for $\sum_{i=1}^{s} \mu_i Y_i$ where $\mu_i \geq 0$ and $\sum_{i=1}^{s} \mu_i = 1$ where $s$ is the number of fuzzy rules. $(A + B) + * = (A + B) + (A + B)^T$. 
II. PRELIMINARIES

Consider a nonlinear T-S fuzzy model obtained from a nonlinear system using techniques [6]:

\[
\dot{x}(t) = A_\mu x(t) + B_\mu u(t) \tag{1}
\]

where \( x = [x_1, \cdots, x_n]^T \) is the state vector, \( u = [u_1, \cdots, u_m]^T \) is the control input. All system matrices \( A_\mu, B_\mu \) are of real constant matrices, depending on the time-varying membership vector \( \mu \) and having appropriate dimensions.

A fuzzy PDC controller is considered here and displayed below

\[
u(t) = \sum_{i=1}^{n} \mu_i K_i x(t) = K_\mu x(t) \tag{2}
\]

where the controller gain matrix \( K_i \in \mathbb{R}^{m \times n} \) is the state feedback gain to be determined.

Substituting (2) into (1) yields a closed-loop fuzzy system

\[
\dot{x}(t) = (A_\mu + B_\mu K_\mu) x(t) = \bar{A}_\mu x(t) \tag{3}
\]

where at any instant the fuzzy system matrices are given by the convex combination of local T-S models and the time-varying parameter vector \( \mu \) belongs to the unit simplex \( \Omega = \{ \mu \in \mathbb{R}^s_+ \text{(positive real)} | \mu_i \geq 0 \text{ and } \sum_{i=1}^{s} \mu_i = 1 \} \).

Recall [10], [11], [12] and references therein where the definition for quadratic stability is given for a forced control system

\[
\dot{x} = (A_\mu + B_\mu K_\mu) x \tag{4}
\]

the forced, disturbance free fuzzy system (4) is said to be quadratically stabilizable by a fuzzy controller if there exists a symmetric matrix \( 0 < P \in \mathbb{R}^{n \times n} \) and a non symmetric matrix gain \( K_\mu \) such that the following parameter-dependent LMIs (PD-LMIs) are satisfied for continuous-time systems:

\[
M_{\mu \mu} = \bar{A}_\mu^T P + P \bar{A}_\mu - (A_\mu + B_\mu K_\mu)^T P + * < 0. \tag{5}
\]

By congruence transformation, \( M_{\mu \mu} < 0 \) is equivalent to

\[
(Q A_\mu^T + F_\mu^T B_\mu^T) + * < 0 \tag{5}
\]

where \( F_\mu = K_\mu Q \) and thus \( K_\mu = F_\mu Q^{-1} \). The result is readily obtained if the quadratic Lyapunov function

\[
V(x) = x^T Q^{-1} x, \quad P = Q^{-1} > 0
\]

is used.

Now the question arises if one wants to extend the quadratic Lyapunov function to polynomial Lyapunov function of the form \( V(x) = x^T Q^{-1}(x) x \). One has to ensure that \( \nabla_x V(x) = 2Q^{-1}(x) x \) is a gradient (vector) function of a positive definite function [13], [?]. This equality \( \nabla_x V(x) = 2Q^{-1}(x) x \) usually does not hold true due to chain rule operation in derivation. Yet, existing paper implements this condition by adding a set of equality constraints into SOS.

Motivated by the result [14], we investigate homogeneous Lyapunov functions and its definition is stated first.

**Definition 1.** [14] A function \( V(s) : \mathbb{R}^n \rightarrow \mathbb{R} \) is said to be a (positive) homogeneous Lyapunov function of degree \( r \), if \( V(x) \) is a Lyapunov function and if

\[
V(\lambda x) = \lambda^r V(x) \tag{6}
\]

holds for all \( x \in \mathbb{R}^n \) and all \( \lambda \geq 0 \).

An appealing property of homogeneous functions is given by Euler’s homogeneity relation (Euler’s identity).

**Theorem 1.** (Euler’s homogeneity relation, [14]) \( V(x) \) is a homogeneous function of degree \( r \), if and only if \( V(x) \) satisfies

\[
rV(x) = x^T \nabla_x V(x) \tag{7}
\]

Differentiating (8) with respect to \( x \) leads to

\[
r \nabla_x V(x) = \nabla_x V(x) + \nabla_{xx} V(x) x \tag{9}
\]

and collecting terms, we have

\[
\nabla_x V(x) = \frac{1}{r-1} \nabla_{xx} V(x) x. \tag{10}
\]

Multiplying the left side of the equation (9) by \( x^T \) yields

\[
r x^T \nabla_x V(x) = x^T \nabla_x V(x) + x^T \nabla_{xx} V(x) x. \tag{11}
\]

and collecting terms take us to

\[
(r-1) x^T \nabla_x V(x) = x^T \nabla_{xx} V(x) x
\]

and together with (7)

\[
V(x) = \frac{1}{r(r-1)} x^T \nabla_{xx} V(x) x. \tag{11}
\]
Another useful relation is deployed below.

**Corollary 1.** Let $V$ be a homogeneous function of degree $r$, then we can derive $\dot{V}$.

$$
\dot{V}(x) = \frac{dV(x)}{dt} = \frac{\partial V(x)}{\partial x_1} \frac{dx_1}{dt} + \frac{\partial V(x)}{\partial x_2} \frac{dx_2}{dt} + \ldots + \frac{\partial V(x)}{\partial x_n} \frac{dx_n}{dt}
$$

$$
= \begin{bmatrix}
\frac{dx_1}{dt} \\
\frac{dx_2}{dt} \\
\vdots \\
\frac{dx_n}{dt}
\end{bmatrix}
$$

$$
= \dot{x}^T \nabla_x V(x)
$$

with the identity (10)

$$
= \dot{x}^T \left( \frac{1}{r-1} \nabla_{xx} V(x) \right)
$$

$$
= \frac{1}{r-1} \dot{x}^T \nabla_{xx} V(x) . \tag{12}
$$

The significances of (11) and (12) are that the homogeneous Lyapunov function can be constructed into quadratic-like form that wraps its Hessian matrix inside. Furthermore, the time derivative of the Lyapunov function can be found easily by replacing $x^T$ with $\dot{x}^T$. Consequently, the relations (11) and (12) establish the foundation of the main results of this paper.

### III. $H\infty$ State Feedback Control

Having established the machinery behind this investigation, we, to apply the analysis shown above, consider the following polynomial fuzzy system using technique [5].

$$
\dot{x}(t) = A(x)x(t) + B_1(x)u(t) + B_2(x)\omega(t) \tag{13}
$$

$$
z(t) = C(x)x(t) + D_\mu u(t) \tag{14}
$$

where $x(t) \in \mathbb{R}^n$ is the state vector, and $u(t) \in \mathbb{R}^m$ is the control input vector. System and input matrices are defined as $A(x) = \sum_{i=1}^s \mu_i A_i(x)$ and $B_1(x) = \sum_{i=1}^s \mu_i B_1_i(x)$ where $A_i(x)$, $B_1_i(x)$ and same for $B_2_i(x)$ are of compatible dimensions and are function of $x$, $\mu_i(x(t))$, $i = 1, 2, \ldots, s$, are the normalized grades of membership and exhibit the following properties: $\mu_i(x(t)) \geq 0 \forall i$, and $\sum_{i=1}^n \mu_i(x(t)) = 1$.

In this paper, a polynomial fuzzy controller has the following form

$$
u(t) = K(x)\dot{x} \tag{20}
$$

where $K(x) = \sum_{i=1}^n \mu_i L_i(x)$. The closed-loop system becomes

$$
\dot{x}(t) = \sum_{i=1}^s \sum_{j=1}^s \mu_i \mu_j (A_i(x) + B_{1i}(x)K_j(x))x(t) + B_{2i}(x)\omega(t). \tag{16}
$$

**Theorem 2.** The closed-loop fuzzy system (16) is stabilizable by the state feedback controller (15), if there exist symmetric matrices $Q(x) = Q^T(x) \in \mathbb{R}^{m\times n}$, and non-symmetric matrices $F_i(x) \in \mathbb{R}^{m\times n}$ such that the following SOS conditions are satisfied.

$$
u^T(Q(x) - \epsilon_1(x)I)v \text{ is SOS}
$$

$$-\nu^T(M_1(x) + \epsilon_2(x)I)v \text{ is SOS}
$$

$$-\nu^T(M_{ij}(x) + M_{ji}(x) + \epsilon_3(x)I)v \text{ is SOS}
$$

where $v \in \mathbb{R}^{n\times n}$ is a vector that is independent of $x$ and $\epsilon_1(x) > 0$, $\epsilon_2(x) > 0$, $\epsilon_3(x) > 0$, are sufficiently small numbers that can be function of $x$ as well.

$$M_{ij} = \begin{bmatrix}
\Phi_{ij} & * & * \\
B_{2i}(x) & -\gamma^2 I & 0 \\
& d_{ij} & 0 & -I
\end{bmatrix}
$$

$$F_i(x) = K_j(x)Q(x)
$$

### IV. $H\infty$ Observed--State Feedback Control

Having established the machinery behind this investigation, we, to apply the analysis shown above, consider the following polynomial fuzzy system using technique [5].

$$\dot{x}(t) = A_\mu(x)x(t) + B_1(x)u(t) + B_2(x)\omega(t) \tag{17}
$$

$$y(t) = C_1(x)x(t) + D_\mu \omega(t) \tag{18}
$$

$$z(t) = C_2(x)x(t) \tag{19}
$$

via estimated-state feedback controller

$$u(t) = K(\hat{x})\dot{\hat{x}} \tag{20}
$$

where $x(t) \in \mathbb{R}^n$ is the state vector and $u(t) \in \mathbb{R}^m$ is the control input vector. System and input matrices are same as the previous section.

In this paper, a polynomial fuzzy observer has the following form

$$\dot{\hat{x}}(t) = A_\mu(x)\dot{\hat{x}}(t) + B_1(x)\hat{\omega}(t) + L_\mu(\hat{x})(y(t) - \hat{y}(t))
$$

$$= A_\mu(x)\dot{\hat{x}}(t) + B_1(x)\hat{\omega}(t) + L_\mu(\hat{x})C_1(x)\epsilon(t)
$$

$$+ L_\mu(\hat{x})D_\mu \omega(t) \tag{21}
$$

$$\hat{y}(t) = C_1(x)\hat{x}(t) \tag{22}
$$

where $L_\mu(\hat{x}) = \sum_{i=1}^n \mu_i L_i(\hat{x})$. 


The error dynamic has the following form

\[
\dot{e}(t) = \dot{x}(t) - \dot{\hat{x}}(t) = (A_\mu(x)x - \dot{\hat{A}}_\mu(\hat{x})\dot{\hat{x}}) + B_{2\mu}(x)\omega(t) - L_\mu(\hat{x})C_{1\mu}e(t) - L_\mu(\hat{x})D_{\mu}\omega(t) + (B_{2\mu}(x) - L_\mu(\hat{x})D_{\mu})\omega(t).
\]  

(23)

**Theorem 3.** The closed-loop fuzzy system (17)-(19) is stabilizable by the estimated-state feedback controller (20), if there exist symmetric matrices \(U(e) = U^T(e) \in R^{n \times n}\), and non-symmetric matrices \(L_i(\hat{x}) \in R^{n \times n}\) such that the following SOS conditions are satisfied.

\[
v^T(Q(x) - \epsilon_1(x)I)v \text{ is SOS}
\]

\[
v^T(U(e) - \epsilon_2(e)I)v \text{ is SOS}
\]

\[
-\bar{v}^T(N_{ii}(\hat{x}, e) + \epsilon_3(\hat{x}, e)I)\bar{v} \text{ is SOS}
\]

\[
-\bar{v}^T(N_{ij}(\hat{x}, e) + N_{ji}(\hat{x}, e) + \epsilon_4(\hat{x}, e)I)\bar{v} \text{ is SOS}
\]

where \(v \in R^{n \times n}\) is a vector that is independent of \(x\) and \(\epsilon_1(e) > 0, \epsilon_2(e) > 0, \epsilon_3(x, e) > 0, \epsilon_4(x, e) > 0\), are sufficiently small numbers that can be function of \(x\) as well.

\[
N_{ij} = \begin{bmatrix}
W_{ij} & * & * & QC_{2\mu}(x) \\
B_{2i}(x) & Y_{ij} & * & 0 \\
C_{2\mu}(x)Q(x) & 0 & 0 & I
\end{bmatrix}
\]

\[
H_j(\hat{x}, e) = U(e)L_j(\hat{x})
\]

\[
\dot{\hat{A}}_i(x, \hat{x}) = A_i(x)x - A_i(\hat{x})\hat{x}.
\]

(24)

**V. ILLUSTRATIVE EXAMPLES**

In this section, examples are demonstrated to illustrate the validity of the design approach. Theorems 2 is applied via SOSTOOLS to show that polynomial Lyapunov matrices \(Q(x)\) exist for illustrated examples.

**Example 1.** Consider the following nonlinear system:

\[
\begin{aligned}
\dot{x}_1 &= -x_1 + x_2^2 - \frac{3}{2}x_1x_2 - \frac{3}{8}x_1^2x_2^2 + \frac{1}{4}x_2 - x_1^2x_2 \\
\dot{x}_2 &= 1.25\omega \\
\dot{\omega} &= -\frac{1}{4}x_2^2 + 1.25\omega \\
\dot{x}_2 &= 1.1u
\end{aligned}
\]

where fuzzy model is

\[
A_1(x) = \begin{bmatrix}
-4 + x_1^2 - \frac{3}{2}x_2 - \frac{7}{4}x_3 - \frac{1}{4}x_2^2 \\
0 \\
0
\end{bmatrix},
\]

\[
A_2(x) = \begin{bmatrix}
2 + \frac{3}{2}x_2^2 - \frac{3}{4}x_2 - \frac{1}{4}x_3^2 \\
0 \\
0
\end{bmatrix},
\]

\[
B_u = \begin{bmatrix}
0 \\
1.1
\end{bmatrix}, \quad B_\omega = \begin{bmatrix}
1.25 \\
0
\end{bmatrix}, \quad C_z = \begin{bmatrix}
0 & 0
\end{bmatrix}
\]

\[
\mu_1(t) = \frac{1}{4}x_1 + \frac{1}{2}, \quad \mu_2(t) = -\frac{1}{4}x_1 + \frac{1}{2}
\]

\[
\omega(t) = \begin{bmatrix}
0.05\sin(0.05t) + \cos(0.05t) \\
0, \quad t > 10
\end{bmatrix}
\]

Solving, theorem 2 yields

\[
Q(x) = \begin{bmatrix}
0.0063x_1^2 + 0.01x_2^2 & 0 \\
0 & 0.01x_1^2 + 0.01529x_2^2
\end{bmatrix}
\]

and the corresponding gains are

\[
K_1(x) = \begin{bmatrix}
-0.425x_1^2 + 0.902x_1 - 1.468 \\
0.81x_2^2 + 0.005x_2 - 5.2662
\end{bmatrix}
\]

\[
K_2(x) = \begin{bmatrix}
0.238x_1^2 - 1.186x_1 + 0.0235 \\
-0.107x_2^2 - 0.001x_2 - 5.4383
\end{bmatrix}
\]

\[
\gamma = 1.4774
\]

**VI. CONCLUSION**

An observed-state feedback synthesis problem is considered for a class of polynomial fuzzy systems. By introducing the homogeneous Lyapunov function of degree \(r\) in \(x\), we remove the two constraints inherited in
the existing SOS methods where (1) $P(\tilde{x})$ only depends on state $\tilde{x}$ and whose corresponding rows in $B(x)$ are zero (the coupling effect) and (2) removing the derivative Lyapunov term. Furthermore, the proposed method provides a more general assumption and computationally tractable SOS-based scheme searching for existence of SOS decomposition, thus leading to characterization of solution to non-quadratic stability regarding to fuzzy polynomial fuzzy control systems.

REFERENCES


